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A new asymptotic model for a composite piezoelastic plate

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Abstract

A new asymptotic homogenization piezoelastic composite plate model is obtained. Derivation is based on a modified two-scale asymptotic homogenization technique applied to a rigorously formulated piezoelectric problem for a three-dimensional thin composite layer of a periodic structure. The obtained model makes it possible to determine both local fields and the effective properties of piezoelectric plate by means of solution of the obtained three-dimensional local unit cell problems and a global two-dimensional piezoelastic problem for a homogenized anisotropic plate. It is shown, in particular that the effective stiffnesses generally depend on the local piezoelectric constants of the material. The general symmetry properties of the effective stiffnesses and piezoelectric coefficients of the homogenized plate are derived. The general model is applied to a practically important case of a laminated anisotropic piezoelastic plate, for which the analytical formulas for the effective stiffnesses, piezoelectric and dielectric coefficients are obtained. Theory is illustrated by a numerical example of a piezoelectric laminated plate of a specific structure.

Keywords: Piezoelastic composite plate; Asymptotic homogenization; Local problems; Effective stiffnesses; Piezoelectric and dielectric properties

1. Introduction

The general homogenization models and their applications for the periodic composite and reinforced structures were developed earlier using the asymptotic homogenization techniques, see Kalamkarov (1987, 1989, 1992), Kalamkarov and Kolpakov (1996, 1997), and Kolpakov (1982). Many problems in the framework of elasticity and thermoelasticity have been solved using these models. The mathematical framework of the asymptotic homogenization technique can be found in Bensoussan et al. (1978) and Sanchez-Palencia (1980). This method is mathematically rigorous, and it enables the prediction of both the local and overall averaged properties of the composite solid.

The homogenization model of periodically inhomogeneous in planar directions elastic plate has been first introduced by Duvaut (1976). It should be noted, however, that the direct application of the

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asymptotic homogenization technique to a two-dimensional plate or shell theory will not provide the satisfactory results if the spatial inhomogeneities of the material vary on a scale comparable with the small thickness of the three-dimensional solid under study. A modified approach developed by Caillerie (1984) consisted in applying the two-scale asymptotic homogenization formalism to three-dimensional problem for a thin inhomogeneous layer. The similar approach was applied by Kohn and Vogelius (1984) to the problem of bending of a thin homogeneous elastic layer with a rapidly varying thickness, see also Lewinski (1992).

Present paper is dealing with the piezoelectric problem for a three-dimensional thin composite solid of a periodic structure on the basis of asymptotic homogenization. Modeling of piezoelectric composites has become nowadays an important problem with the emerging area of smart structures based in many cases on the use of piezoelectric actuators and sensors. A survey by Rao and Sunar (1994) has demonstrated the wide and important applications of piezoelectric materials in many areas of science and engineering. The use of piezoelectric actuators and sensors as elements of smart structures was investigated by Crawley and de Luis (1987), Reddy (1999), Ashida and Tauchert (1998), Kalamkarov and Drozdov (1997), Tzou (1993), Tzou and Bao (1995), Wang and Rogers (1991) and Zhou and Tiersten (1994) among others. A large number of analytical results has been published, concerning different aspects of piezoelectrical problems, such as vibrations of plates and shells (Tiersten, 1969; Batra et al., 1996; Librescu et al., 1996, 1997; Ding et al., 1997); shape control (Koconis et al., 1994a,b); cracks (Pak, 1990). Piezoelectric composite plates and laminates were analyzed by Lee (1990), Lee and Jiang (1996), Bisegna and Maceri (1996a,b); Sosa (1992) and Heyliger (1994, 1997). Problems in thermopiezoelasticity with relevance to smart composite structures were considered by Tauchert et al. (1999). Piezothermoelastic problems for composite plates and beams have been studied by Blandford et al. (1999) and Jonnalagadda et al. (1993, 1994). Asymptotic approaches for thin piezoelectric plates were developed by Maugin and Attou (1990) and Cheng et al. (2000).

In the present paper a new asymptotic homogenization piezoelastic composite plate model is obtained. Derivation is based on a modified two-scale asymptotic homogenization techniques, see Kalamkarov (1992), applied to a rigorously formulated piezoelectric problem for a three-dimensional thin composite layer of a periodic structure. This model makes it possible to determine both local fields and the effective properties. The effective stiffnesses, piezoelectric and dielectric coefficients of the homogenized piezoelastic plate can be calculated by means of solution of the obtained set of three-dimensional local unit cell problems. And local mechanical and electrical fields can be determined using the solutions of local unit cell problems and the solution of the global two-dimensional problem for a homogenized anisotropic plate.

Following this introduction, the basic relations of the three-dimensional piezoelectric problem are formulated in the Section 2. The two-scale asymptotic expansions are introduced in Section 3. The basic relations of the homogenized piezoelastic composite plate model are derived in Sections 4 and 5. Section 6 deals with the basic symmetry properties of the effective coefficients of the homogenized piezoelectric plate. The general model is applied to a particular case of laminated piezoelastic plate in Section 7. The analytical formulas for the effective stiffnesses, piezoelectric and dielectric coefficients of a laminated anisotropic piezoelastic plate are obtained. Theory is illustrated by a numerical example of a three-layer piezoelectric laminated plate in Section 8. Finally, Section 9 concludes the paper.

2. Problem formulation

Let us consider a thin three-dimensional composite layer of a periodic structure obtained by repeating a certain small unit cell P_ε in the Ox_1x_2-plane, see Fig. 1. Here ε is a characteristic dimension of a periodicity cell, which is assumed to be small as compared with the tangential dimensions of the solid in whole, that is formalized in the form $\varepsilon \to 0$. As a result, we obtain a solid of a periodic structure occupying the domain Q_ε with the small thickness, see Fig. 1. Note that the shape of lateral surface of the layer S_ε is determined by the

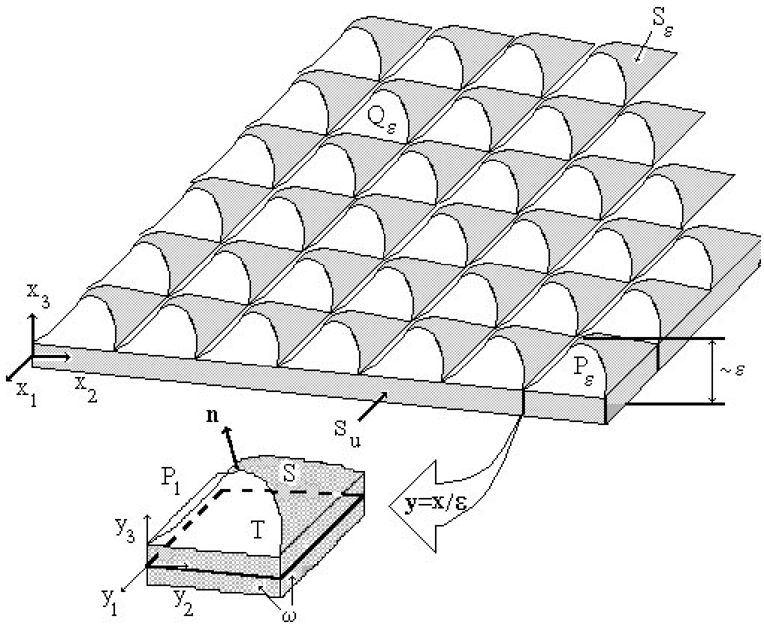

Fig. 1. Thin composite solid of a periodic structure and its periodicity cell.

type of surface reinforcement, for example by shape of stiffeners or reinforcing ribs. In particular, this surface can be plane if surface reinforcements are not used. As $\varepsilon \to 0$, the three-dimensional domain Q_ε converges to a two-dimensional plate-like domain D in the Ox_1x_2-plane.

We will analyze a piezoelastic problem for the above-described three-dimensional composite solid. The problem formulation consists of the following equilibrium equations and equation of electrostatics:

$$\sigma^{\varepsilon}_{ij,j} = \varepsilon^{-3} f_i \quad \text{in } Q_\varepsilon \tag{2.1}$$

$$E^{\varepsilon}_{i,i} = 0 \quad \text{in } Q_\varepsilon \tag{2.2}$$

where $\sigma^{\varepsilon}_{ij}$ are stresses, E^{ε}_{i} is electric field.

The boundary conditions are given as follows:

$$\mathbf{u}^{\varepsilon} = 0, \quad F^{\varepsilon} = e(x_1, x_2) \qquad \text{on } S_u \tag{2.3}$$

$$\sigma^{\varepsilon}_{ij} n_j = \varepsilon^{-3} g_i, \quad E^{\varepsilon}_{i} n_i = 0 \qquad \text{on } S_\varepsilon \tag{2.4}$$

Here $\mathbf{u}^{\varepsilon}$ are displacements, F^{ε} is electric potential.

It is assumed that the solid is clamped at the boundary surface S_u, see Fig. 1. The second condition in Eq. (2.3) means that voltage is applied only at the boundary surface S_u.

The local constitutive equations can be written as follows (see e.g., Kalamkarov (1992)):

$$\sigma^{\varepsilon}_{ij} = \varepsilon^{-3}(a_{ijkl}(\mathbf{y})\,\partial u^{\varepsilon}_k/\partial x_1 - e_{kij}(\mathbf{y})\,\partial F^{\varepsilon}/\partial x_k) \tag{2.5}$$

$$E^{\varepsilon}_{i} = \varepsilon^{-1}(e_{ikl}(\mathbf{y})\,\partial u^{\varepsilon}_k/\partial x_1 + \varepsilon_{ik}(\mathbf{y})\,\partial F^{\varepsilon}/\partial x_k) \tag{2.6}$$

Here the elastic coefficients $a_{ijkl}(\mathbf{y})$, piezoelectric coefficients $e_{kij}(y)$ and dielectric coefficients $\varepsilon_{ik}(y)$ are the functions in so-called fast variables $\mathbf{y} = (y_1, y_2, y_3)$ where $y_k = x_k, /\varepsilon$, $k = 1, 2, 3$, and all these functions are assumed to be periodic in tangential coordinates x_1, x_2, with the above-defined unit cell P_ε. The dependency of these material parameter functions on coordinates x_1, x_2, x_3 within the unit cell P_ε is defined by the spatial inhomogeneity of the composite material under consideration. The above assumption for the material parameter functions is similar to one that was previously introduced in modeling of thin composite layer in framework of elasticity and thermoelasticity, see Kalamkarov (1987, 1989, 1992).

The factor ε^{-3} in Eq. (2.5) assures that effective bending stiffnesses of the plate will be non-zero, as $\varepsilon \to 0$.

3. Asymptotic analysis

In this Section the asymptotic expansions are presented for the functions forming the above-formulated piezoelastic problem (2.1)–(2.6). With the use of two-scale asymptotic expansions, functions in the fast variables $\mathbf{y} = \mathbf{x}/\varepsilon$ and in the slow variable $\mathbf{X} = (x_1, x_2) \in D$ are introduced, see Kalamkarov (1992) and Kalamkarov and Kolpakov (1997).

We will study global deformation of the plate as $\varepsilon \to 0$. In order to do this, we use the following asymptotic expansions:

$$\mathbf{u}^\varepsilon = \varepsilon^{-1}\mathbf{u}^{(-1)}(\mathbf{X}) + \mathbf{u}^{(0)}(\mathbf{X}, \mathbf{y}) + \cdots = \varepsilon^{-1}\mathbf{u}^{(0)}(\mathbf{X}) + \sum_{k=0}^{\infty} \varepsilon^k \mathbf{u}^{(k)}(\mathbf{X}, \mathbf{y}) \tag{3.1}$$

$$E^\varepsilon = \varepsilon^{-1}E^{(-1)}(\mathbf{X}) + E^{(0)}(\mathbf{X}, \mathbf{y}) + \cdots = \varepsilon^{-1}E^{(0)}(\mathbf{X}) + \sum_{k=0}^{\infty} \varepsilon^k E^{(k)}(\mathbf{X}, \mathbf{y}) \tag{3.2}$$

$$\sigma_{ij}^\varepsilon = \sum_{p=-4}^{\infty} \varepsilon^p \sigma_{ij}^{(p)}(x_1, x_2, \mathbf{y}) \tag{3.3}$$

$$E_i^\varepsilon = \sum_{p=-2}^{\infty} \varepsilon^p E_i^{(p)}(x_1, x_2, \mathbf{y}) \tag{3.4}$$

The functions in the right-hand side of Eqs. (3.1)–(3.4) are assumed to be periodic in (y_1, y_2) with periodicity cell T, where T is a projection of the three-dimensional unit cell $P_1 = \{\mathbf{y} = \mathbf{x}/\varepsilon : \ x \in P_\varepsilon\}$ onto the Oy_1y_2-plane, see Fig. 1.

Note that the above expansions (3.1) and (3.3) are different from the expansions used earlier in Caillerie (1984) and Galka et al. (1992). The expansion for the displacements (3.1) is starting with the term of the order of ε^{-1}. The expansion for stresses (3.3) is starting with the term of the order of ε^{-4} in accordance with the expansion for the displacements and the local constitutive equations (2.5).

The following rule of differentiation is assumed for a function $f(\mathbf{X}, \mathbf{y})$ in the arguments $\mathbf{X} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2, y_3)$, such as in the right-hand sides of Eq. (3.1):

$$\partial f/\partial x_3 = \varepsilon^{-1} f_{,3y}, \qquad \partial f/\partial x_\alpha = \varepsilon^{-1} f_{,\alpha y} + f_{,\alpha x} \quad (\alpha = 1, 2) \tag{3.5}$$

Here and in the sequel all Latin indices assume values 1, 2, 3 and all Greek indices assume values 1, 2; $,\alpha x$ denotes $\partial/\partial x_\alpha$ and, jy denotes $\partial/\partial y_j$; $\delta_{\alpha 3} = 0$.

Substituting expansions (3.1)–(3.4) into the local constitutive equations (2.5) and (2.6) and representing the derivatives of the test function in accordance with Eq. (3.5), we obtain

$$\sum_{p=-4}^{\infty} \varepsilon^p \sigma_{ij}^{(p)} = \sum_{s=-1}^{\infty} \varepsilon^{s-3} \left[a_{ijk\alpha}(\mathbf{y}) u_{k,\alpha x}^{(s)} + \varepsilon^{-1} a_{ijkl}(\mathbf{y}) u_{k,ly}^{(s)} - e_{\alpha ij}(\mathbf{y}) F_{,\alpha x}^{(s)} - \varepsilon^{-1} e_{kij}(\mathbf{y}) F_{,ky}^{(s)} \right] \tag{3.6}$$

$$\sum_{p=-2}^{\infty}\varepsilon^p E_i^{(p)} = \sum_{s=-1}^{\infty}\varepsilon^{s-1}\Big[e_{ik\alpha}(\mathbf{y})u^{(s)}_{k,\alpha x} + \varepsilon^{-1}\boldsymbol{e}_{ikl}(\mathbf{y})u^{(s)}_{k,ly} + \varepsilon_{i\alpha}(\mathbf{y})F^{(s)}_{,\alpha x} + \varepsilon^{-1}\varepsilon_{ij}(\mathbf{y})F^{(s)}_{,jy}\Big] \tag{3.7}$$

Equating the terms with the same powers of ε in Eqs. (3.6) and (3.7), and taking into account that $\mathbf{u}^{(-1)}$ and $E^{(-1)}$ do not depend on $\mathbf{y}$, we obtain

$$\varepsilon^{-4} : \sigma_{ij}^{-4} = a_{ijk\alpha}(\mathbf{y})u^{(-1)}_{k,\alpha x} + a_{ijkl}(\mathbf{y})u^{(0)}_{k,ly} - e_{\alpha ij}(\mathbf{y})F^{(-1)}_{,\alpha x} - e_{kij}(\mathbf{y})F^{(0)}_{,ky}] \tag{3.8}$$

$$\varepsilon^{-3} : \sigma_{ij}^{-3} = a_{ijk\alpha}(\mathbf{y})u^{(0)}_{k,\alpha x} + a_{ijkl}(\mathbf{y})u^{(1)}_{k,ly} - e_{\alpha ij}(\mathbf{y})F^{(0)}_{,\alpha x} - e_{kij}(\mathbf{y})F^{(1)}_{,ky}] \tag{3.9}$$

and so on.

$$\varepsilon^{-2} : E_i^{(-1)} = e_{ik\alpha}(\mathbf{y})u^{(-1)}_{k,\alpha x} + \boldsymbol{e}_{ikl}(\mathbf{y})u^{(0)}_{k,ly} + \varepsilon_{i\alpha}(\mathbf{y})F^{(-1)}_{,\alpha x} + \varepsilon_{ij}(\mathbf{y})F^{(0)}_{,jy}] \tag{3.10}$$

$$\varepsilon^{-1} : E_i^{(0)} = e_{ik\alpha}(\mathbf{y})u^{(0)}_{k,\alpha x} + \boldsymbol{e}_{ikl}(\mathbf{y})u^{(1)}_{k,ly} + \varepsilon_{i\alpha}(\mathbf{y})F^{(0)}_{,\alpha x} + \varepsilon_{ij}(\mathbf{y})F^{(1)}_{,jy}] \tag{3.11}$$

and so on.

4. Equations of balance and boundary conditions for the two-dimensional plate

Substituting expansions (3.3) and (3.4) into Eqs. (2.1) and (2.2), and equating the terms with identical powers of ε, we obtain

$$\sigma^{(p)}_{ij,jy} + \sigma^{(p-1)}_{i\alpha,\alpha x} = 0 \quad \text{for } p = -4, -2, \ldots; \qquad \sigma^{(-3)}_{ij,jy} + \sigma^{(-4)}_{i\alpha,\alpha x} = f_i \quad \text{in } P_1 \tag{4.1}$$

$$\sigma^{(p)}_{ij} n_j = 0 \quad \text{for } p = -4, -2, \ldots; \qquad \sigma^{(-3)}_{ij} n_j = g_i \quad \text{on } S \tag{4.2}$$

$$E^{(p)}_{i,iy} + E^{(p-1)}_{\alpha,\alpha x} = 0 \quad \text{in } P_1 \quad \text{for } p = -1, 0 \tag{4.3}$$

$$E^{(p)}_i n_i = 0 \quad \text{on } S \;\; \text{for } p = -1, 0, \ldots \tag{4.4}$$

Here S is the lateral surface of the periodicity cell P_1, see Fig. 1.

Averaging Eqs. (4.1) and (4.2) as well as Eq. (4.1) multiplied by y_3, and taking into account that $\langle f_{,iy}\rangle = 0$ for any periodic in y_1, y_2 function, and having normal derivative equal to zero on S, we obtain the following equations for force resultants $N^{(p)}_{ij} = \langle\sigma^{(p)}_{ij}\rangle$, moment resultants $M^{(p)}_{\alpha\beta} = -\langle\sigma^{(p)}_{\alpha\beta} y_3\rangle$ and averaged electric field $E^{(p)}_{\beta} = \langle E^{(p)}_{\beta}\rangle$ $(i, j = 1, 2, 3; \alpha, \beta = 1, 2)$:

$$N^{(p)}_{i\beta,\beta x} = 0, \quad p = -4, -2, \ldots; \;\; N^{(-3)}_{i\beta,\beta x} = \langle f_i\rangle + \langle g_i\rangle \tag{4.5}$$

$$N^{(-2)}_{3\beta,\beta x} = \langle f_3\rangle + \langle g_3\rangle \tag{4.6}$$

$$M^{(p-1)}_{i\beta,\beta x} - N^{(p)}_{i3} = 0, \tag{4.7}$$

$$E^{(p)}_{\beta,\beta x} = 0, \quad p = -1, 0, \tag{4.8}$$

Here

$$\langle\cdot\rangle = (\operatorname{meas} T)^{-1}\int_{P_1}(\cdot)\,\mathrm{d}y \quad \text{and} \quad \langle\cdot\rangle s = (\operatorname{meas} T)^{-1}\int_{S}(\cdot)\,\mathrm{d}y$$

are the average values over the periodicity cell P_1 and its lateral surface S.
The following relations take place as $\varepsilon \to 0$

$$\varepsilon^{-1}\int_{Q_\varepsilon} f(\mathbf{X},\mathbf{x}/\varepsilon)\,\mathrm{d}\mathbf{x} \to \int_D \langle f\rangle(\mathbf{X})\,\mathrm{d}\mathbf{X},\quad \varepsilon^{-1}\int_{S_\varepsilon} f(\mathbf{X},\mathbf{x}/\varepsilon)\,\mathrm{d}\mathbf{x} \to \int_D \langle f\rangle_s(\mathbf{X})\,\mathrm{d}\mathbf{X}$$

Substituting the expansions (3.1) and (3.2) into the original boundary conditions (2.3) and (2.4) and equating the terms with the same powers of ε, we obtain

$$\mathbf{u}^{(k)} = 0,\quad k=-1,0,\ldots;\qquad E^{(k)}=0,\quad k=-1,1,\ldots;\qquad E^{(k)} = e(\mathbf{X})\quad \text{on } \partial D \tag{4.9}$$

5. Constitutive relations for the two-dimensional homogenized plate model

From Eqs. (4.1) and (4.2), for $p=-4$, and Eqs. (4.3) and (4.4) for $p=-1$ we obtain the following equations:

$$\left(a_{ijkl}(\mathbf{y})u^{(0)}_{k,ly} - e_{kij}F^{(0)}_{,ky} + a_{ijk\alpha}(\mathbf{y})u^{(-1)}_{k,\alpha x} - e_{\alpha ij}F^{(-1)}_{,\alpha x}\right)_{,jy} = 0 \quad \text{in } P_1 \tag{5.1}$$

$$\left(e_{ikl}(\mathbf{y})u^{(0)}_{k,ly} + \varepsilon_{ij}F^{(0)}_{,jy} + e_{ik\alpha}(\mathbf{y})u^{(-1)}_{k,\alpha x} + \varepsilon_{i\alpha}F^{(-1)}_{,\alpha x}\right)_{,jy} = 0 \quad \text{in } P_1 \tag{5.2}$$

with the boundary conditions

$$\left(a_{ijkl}(\mathbf{y})u^{(0)}_{k,ly} - e_{kij}F^{(0)}_{,ky} + a_{ijk\alpha}(\mathbf{y})u^{(-1)}_{k,\alpha x} - e_{\alpha ij}F^{(-1)}_{,\alpha x}\right)n_j = 0 \quad \text{on } S \tag{5.3}$$

$$\left(e_{ikl}(\mathbf{y})u^{(0)}_{k,ly} + \varepsilon_{ij}F^{(0)}_{,jy} + e_{ik\alpha}(\mathbf{y})u^{(-1)}_{k,\alpha x} + \varepsilon_{i\alpha}F^{(-1)}_{,\alpha x}\right)n_j = 0 \quad \text{on } S \tag{5.4}$$

To solve Eqs. (5.1)–(5.4), we introduce the set of so-called local unit cell problems. The unit cell problem corresponding to average deformation $\delta_{mk}\delta_{n\alpha}$ is the following:

$$\begin{aligned}&\left(a_{ijkl}(\mathbf{y})\mathrm{N}^{0m\alpha}_{k,ly} - e_{kij}\Phi^{0m\alpha}_{,ky} + a_{ijm\alpha}(\mathbf{y})\right)_{,jy} = 0 \quad \text{in } P_1\\ &\left(e_{ikl}(\mathbf{y})\mathrm{N}^{0m\alpha}_{k,ly} + \varepsilon_{ij}\Phi^{0m\alpha}_{,jy} + e_{im\alpha}(\mathbf{y})\right)_{,jy} = 0 \quad \text{in } P_1\end{aligned} \tag{5.5}$$

with boundary conditions

$$\begin{aligned}&\left(a_{ijkl}(\mathbf{y})\mathrm{N}^{0m\alpha}_{k,ly} - e_{kij}\Phi^{0m\alpha}_{,ky} + a_{ijm\alpha}(\mathbf{y})\right)n_j = 0 \quad \text{on } S\\ &\left(e_{ikl}(\mathbf{y})\mathrm{N}^{0m\alpha}_{k,ly} + \varepsilon_{ij}\Phi^{0m\alpha}_{,jy} + e_{im\alpha}(\mathbf{y})\right)n_j = 0 \quad \text{on } S\end{aligned} \tag{5.6}$$

The unit cell problem corresponding to average electric field $\delta_{k\alpha}$ is the following:

$$\begin{aligned}&\left(a_{ijkl}(\mathbf{y})N^{\alpha}_{k,ly} - e_{kij}\Phi^{\alpha}_{,ky} - e_{\alpha ij}\right)_{,jy} = 0 \quad \text{in } P_1\\ &\left(e_{ikl}(\mathbf{y})N^{\alpha}_{k,ly} + \varepsilon_{ij}\Phi^{\alpha}_{,jy} + \varepsilon_{i\alpha}\right)_{,jy} = 0 \quad \text{in } P_1\end{aligned} \tag{5.7}$$

with boundary conditions

$$\begin{aligned}&\left(a_{ijkl}(\mathbf{y})N^{\alpha}_{k,ly} - e_{kij}\Phi^{\alpha}_{,ky} - e_{\alpha ij}\right)n_j = 0 \quad \text{on } S\\&\left(e_{ikl}(\mathbf{y})N^{\alpha}_{k,ly} + \varepsilon_{ij}\Phi^{\alpha}_{,jy} + \varepsilon_{i\alpha}\right)n_j = 0 \quad \text{on } S\end{aligned} \tag{5.8}$$

In the analysis of homogenized plate model the possibility of finding an exact solution of the unit cell problem plays an important role, see Kalamkarov (1992) for the elastic case. In the piezoelastic case under consideration we can find an exact solution of unit cell problems (5.5) and (5.6) for $k = 3$. This solution can be written as follows:

$$\mathbf{N}^{03\alpha} = -y_3\mathbf{e}_\alpha, \qquad \Phi^{03\alpha} = 0 \tag{5.9}$$

The solution (5.9) can be verified by substituting formula (5.9) into Eqs. (5.5) and (5.6) and taking into account that $a_{ij\alpha3}(\mathbf{y}) = a_{ij3\alpha}(\mathbf{y})$ and $e_{i\alpha3}(\mathbf{y}) = e_{i3\alpha}(\mathbf{y})$.

Using Eq. (5.9), we can obtain the solution of the problem (5.1)–(5.4) in the following form:

$$\mathbf{u}^{(0)} = -y_3\mathbf{e}_\alpha u^{(-1)}_{3,\alpha x}(\mathbf{X}) + \mathbf{N}^{0\beta\alpha}(\mathbf{y})u^{(-1)}_{\beta,\alpha x}(\mathbf{X}) - \mathbf{N}^{\alpha}(\mathbf{y})F^{(-1)}_{,\alpha x}(\mathbf{X}) + \mathbf{V}(\mathbf{X}) \tag{5.10}$$

$$F^{(0)} = \Phi^{\alpha}(\mathbf{y})F^{(-1)}_{,\alpha x}(\mathbf{X}) + \Theta(\mathbf{X}) \tag{5.11}$$

From Eqs. (3.8), (3.10) and (5.10) and (5.11) we obtain

$$\sigma^{(-4)}_{ij} = \left(a_{ij\alpha\beta}(\mathbf{y}) + a_{ijkl}(\mathbf{y})\mathrm{N}^{0\beta\alpha}_{k,ly}\right)u^{(-1)}_{\alpha,\beta x} + \left(a_{ijkl}(\mathbf{y})\mathrm{N}^{\alpha}_{k,ly} - e_{\alpha ij} + e_{kij}\Phi^{\alpha}_{,ky}\right)F^{(-1)}_{,\alpha x} \tag{5.12}$$

$$E^{(-1)}_{i} = \left(e_{i\alpha\beta}(\mathbf{y}) + e_{ikl}(\mathbf{y})\mathrm{N}^{0\beta\alpha}_{k,ly}\right)u^{(-1)}_{\alpha,\beta x} + \left(\varepsilon_{i\alpha} + \varepsilon_{ij}\Phi^{\alpha}_{,jy}\right)F^{(-1)}_{,\alpha x} \tag{5.13}$$

Averaging Eqs. (5.12) and (5.13), we obtain

$$N^{(-4)}_{ij} = A_{ij\alpha\beta}u^{(-1)}_{\alpha,\beta x} + E^{n}_{\alpha ij}F^{(-1)}_{,\alpha x} \tag{5.14}$$

$$E^{(-1)}_{i} = E_{i\beta\alpha}u^{(-1)}_{k,\alpha x} + E^{e}_{i\alpha}F^{(-1)}_{,\alpha x} \tag{5.15}$$

where

$$\begin{aligned}&A_{ij\alpha\beta} = \langle a_{ij\alpha\beta}(\mathbf{y}) + a_{ijkl}(\mathbf{y})\mathrm{N}^{0\beta\alpha}_{k,ly}\rangle; \qquad E^{n}_{\alpha ij} = \langle a_{ijkl}(\mathbf{y})\mathrm{N}^{\alpha}_{k,ly} - e_{\alpha ij} + e_{kij}\Phi^{\alpha}_{,ky}\rangle\\&E^{e}_{\alpha ij} = \langle e_{i\alpha\beta}(\mathbf{y}) + e_{ikl}(\mathbf{y})\mathrm{N}^{0\beta\alpha}_{k,ly}\rangle; \qquad E_{i\alpha} = \langle \varepsilon_{i\alpha} + \varepsilon_{ij}\Phi^{\alpha}_{,jy}\rangle\end{aligned} \tag{5.16}$$

Relation (5.16) provide formulas for calculation of effective elastic and piezoelectric constants of the two-dimensional homogenized plate. It is interesting to note that as it is seen from Eq. (5.16) the effective elastic constants $A_{ij\alpha\beta}$ depend on solution $N^{0\beta\alpha}$ of the piezoelectric unit cell problems (5.5) and (5.6). As a result, these elastic constants depend on the local piezoelectric constants.

Solution of Eq. (4.5) for $p = -4$, Eq. (4.7) for $p = -1$ with Eqs. (5.14) and (5.15) and boundary conditions (4.8) for $k = -1$ is

$$u^{(-1)}_{\alpha} = 0; \qquad F^{(-1)}_{\alpha} = 0$$

Therefore, Eqs. (5.10) and (5.11) can be represented as follows:

$$\mathbf{u}^{(0)} = -y_3\mathbf{e}_\alpha u^{(-1)}_{3,\alpha x}(\mathbf{X}) + \mathbf{V}(\mathbf{X}) \tag{5.17}$$

$$F^{(0)} = \Theta(\mathbf{X}) \tag{5.18}$$

It follows from Eq. (5.13) that $\sigma^{(-4)}_{ij} = 0$.

Substituting Eqs. (5.17) and (5.18) into Eqs. (3.9) and (3.11), we obtain

$$\sigma_{ij}^{(-3)} = a_{ijkl}(\mathbf{y})u_{k,ly}^{(1)} - a_{ij\beta\alpha}(\mathbf{y})y_3 u_{3,\alpha x\beta x}^{(-1)} + a_{ij\beta\alpha}(\mathbf{y})V_{\alpha,\beta x} - a_{ijkl}(\mathbf{y}) - u_{k,ly}^{(1)} - e_{kij}F_{,ky}^{(1)} - e_{\alpha ij}\Theta_{,\alpha x} \tag{5.19}$$

$$E_i^{(0)} = \varepsilon_{ij}F_{,jy}^{(1)} + \varepsilon_{i\alpha}\Theta_{,\alpha x} - e_{i\beta\alpha}(\mathbf{y})y_3 u_{3,\alpha x\beta x}^{(-1)} + e_{i\beta\alpha}(\mathbf{y})V_{\alpha,\beta x} + a_{ikl}(\mathbf{y})u_{k,ly}^{(1)} \tag{5.20}$$

Solution of problems (4.1) and (4.2) for $p = -3$, and Eqs. (4.3) and (4.4) for $p = -1$ has the following form:

$$\mathbf{u}^{(1)} = \mathbf{N}^{0\beta\alpha}(\mathbf{y})V_{\alpha,\beta x}(\mathbf{X}) + \mathbf{N}^{1\beta\alpha}(\mathbf{y})u_{3,\alpha x\beta x}^{(-1)}(\mathbf{X}) + \mathbf{N}^{\alpha}(\mathbf{y})\Theta_{,\alpha x}(\mathbf{X}) + \mathbf{W}(\mathbf{X}) \tag{5.21}$$

$$F^{(0)} = \Phi^{\alpha}(\mathbf{y})\Theta_{,\alpha x}(\mathbf{X}) + \Phi^{0\alpha\beta}(\mathbf{y})V_{\alpha,\beta x}(\mathbf{X}) + \Phi^{1\alpha\beta}(\mathbf{y})u_{3,\alpha x\beta x}^{(-1)}(\mathbf{X}) + t(\mathbf{X}) \tag{5.22}$$

where $(\mathbf{N}^{1\beta\alpha}, \Phi^{1\beta\alpha})$ is a solution of the following unit cell problem corresponding to bending of a plate:

$$\begin{aligned} &\left(a_{ijkl}(\mathbf{y})\mathrm{N}_{k,ly}^{1m\alpha} - e_{kij}\Phi_{,ky}^{1m\alpha} - a_{ijm\alpha}(\mathbf{y})y_3\right)_{,jy} = 0 \quad \text{in } P_1 \\ &\left(e_{ikl}(\mathbf{y})\mathrm{N}_{k,ly}^{0m\alpha} - \varepsilon_{ij}\Phi_{,jy}^{1m\alpha} - e_{im\alpha}(\mathbf{y})y_3\right)_{,jy} = 0 \quad \text{in } P_1 \end{aligned} \tag{5.23}$$

with boundary conditions

$$\begin{aligned} &\left(a_{ijkl}(\mathbf{y})\mathrm{N}_{k,ly}^{1m\alpha} - e_{kij}\Phi_{,ky}^{1m\alpha} - a_{ijm\alpha}(\mathbf{y})y_3\right)n_j = 0 \quad \text{on } S \\ &\left(e_{ikl}(\mathbf{y})\mathrm{N}_{k,ly}^{1m\alpha} - \varepsilon_{ij}\Phi_{,jy}^{1m\alpha} - e_{im\alpha}(\mathbf{y})\right)n_j = 0 \quad \text{on } S \end{aligned} \tag{5.24}$$

Substituting Eqs. (5.21) and (5.22) into Eqs. (5.19) and (5.20), we obtain

$$\begin{aligned} \sigma_{ij}^{(-3)} = {} & a_{ijkl}(\mathbf{y})\mathrm{N}_{k,ly}^{0\alpha\beta}(\mathbf{y})V_{\alpha,\beta x}(\mathbf{X}) + a_{ijkl}(\mathbf{y})\mathrm{N}_{k,ly}^{1\alpha\beta}(\mathbf{y})u_{3,\alpha x\beta x}^{(-1)}(\mathbf{X}) + a_{ijkl}(\mathbf{y})\mathbf{N}_{k,ly}^{\alpha}(\mathbf{y})\Theta_{,\alpha x}(\mathbf{X}) \\ & - a_{ij\beta\alpha}(\mathbf{y})y_3 u_{3,\alpha x\beta x}^{(-1)} + a_{ij\beta\alpha}(\mathbf{y})V_{\alpha,\beta x} - e_{kij}(\mathbf{y})\Phi_{,ky}^{\alpha}(\mathbf{y})\Theta_{,\alpha x}(\mathbf{X}) - e_{kij}(\mathbf{y})\Phi_{,ky}^{0k\alpha}(\mathbf{y})V_{\alpha,\beta x}(\mathbf{X}) \\ & - e_{kij}\Phi_{,ky}^{1k\alpha}(\mathbf{y})u_{3,\alpha x\beta x}^{(-1)}(\mathbf{X}) - e_{\alpha ij}(\mathbf{y})\Theta_{,\alpha x} \end{aligned} \tag{5.25}$$

$$\begin{aligned} E_i^{(0)} = {} & \varepsilon_{ij}(\mathbf{y})\Phi_{,jy}^{\alpha}(\mathbf{y})\Theta_{,\alpha x}(\mathbf{X}) + \varepsilon_{ij}(\mathbf{y})\Phi_{,ky}^{0\alpha\beta}(\mathbf{y})V_{\alpha,\beta x}(\mathbf{X}) + \varepsilon_{ij}(\mathbf{y})\Phi_{,ky}^{1\alpha\beta}(\mathbf{y})u_{3,\alpha x\beta x}^{(-1)}(\mathbf{X}) + \varepsilon_{i\alpha}\Theta_{,\alpha x} \\ & - e_{i\beta\alpha}(\mathbf{y})y_3 u_{3,\alpha x\beta x}^{(-1)} + e_{i\beta\alpha}(\mathbf{y})V_{\alpha,\beta x} + e_{ikl}(\mathbf{y})N_{k,ly}^{0\alpha\beta}(\mathbf{y})V_{\alpha,\beta x}(\mathbf{X}) + e_{ikl}(\mathbf{y})\mathbf{N}_{k,ly}^{1\alpha\beta}(\mathbf{y})u_{3,\alpha x\beta x}^{(-1)}(\mathbf{X}) \\ & + e_{ikl}(\mathbf{y})N_{k,ly}^{\alpha}(\mathbf{y})\Theta_{,\alpha x}(\mathbf{X}) \end{aligned} \tag{5.26}$$

Averaging Eqs. (5.25) and (5.26), we obtain after some transformation

$$\begin{aligned} \mathbf{N}_{ij}^{(-3)} = {} & \left\langle a_{ijkl}(\mathbf{y})\mathbf{N}_{k,ly}^{0\alpha\beta}(\mathbf{y}) + a_{ij\beta\alpha}(\mathbf{y}) - e_{kij}(\mathbf{y})\Phi_{,ky}^{0\alpha\beta}(\mathbf{y})\right\rangle V_{\alpha,\beta x}(\mathbf{X}) + \left\langle a_{ijkl}(\mathbf{y})\mathbf{N}_{k,ly}^{1\alpha\beta}(\mathbf{y}) - a_{ij\beta\alpha}(\mathbf{y})y_3 \right. \\ & \left. - e_{kij}\Phi_{,ky}^{1\alpha\beta}(\mathbf{y})\right\rangle u_{3,\alpha x\beta x}^{(-1)}(\mathbf{X}) + \left\langle a_{ijkl}(\mathbf{y})N_{k,ly}^{\alpha}(\mathbf{y}) - e_{kij}(\mathbf{y})\Phi_{,ky}^{\alpha}(\mathbf{y}) - e_{\alpha ij}(\mathbf{y})\right\rangle \Theta_{,\alpha x}(\mathbf{X}) \end{aligned} \tag{5.27}$$

$$\begin{aligned} E_i^{(0)} = {} & \left\langle \varepsilon_{ik}(\mathbf{y})\Phi_{,ky}^{0k\alpha}(\mathbf{y}) + e_{i\beta\alpha}(\mathbf{y}) + e_{ikl}(\mathbf{y})\mathbf{N}_{k,ly}^{0\alpha\beta}(\mathbf{y})\right\rangle V_{\alpha,\beta x}(\mathbf{X}) + \left\langle \varepsilon_{ik}(\mathbf{y})\Phi_{,ky}^{1\alpha\beta}(\mathbf{y}) - e_{i\beta\alpha}(\mathbf{y})y_3 \right. \\ & \left. + e_{ikl}(\mathbf{y})N_{k,ly}^{1\alpha\beta}(\mathbf{y})\right\rangle u_{3,\alpha x\beta x}^{(-1)}(\mathbf{X}) + \left\langle \varepsilon_{ij}(\mathbf{y})\Phi_{,jy}^{\alpha}(\mathbf{y}) + \varepsilon_{i\alpha} + e_{ikl}(\mathbf{y})\mathbf{N}_{k,ly}^{\alpha}(\mathbf{y})\right\rangle \Theta_{,\alpha x}(\mathbf{X}) \end{aligned} \tag{5.28}$$

Multiplying Eq. (5.25) by $(-y_3)$ and averaging, we obtain

$$
\begin{aligned}
M_{ij}^{(-3)} = & -\left\langle y_3\left(a_{ijkl}(\mathbf{y})\mathrm{N}_{k,ly}^{0\alpha\beta}(\mathbf{y}) + a_{ij\beta\alpha}(\mathbf{y}) - e_{kij}(\mathbf{y})\Phi_{,ky}^{0\alpha\beta}(\mathbf{y})\right)\right\rangle V_{\alpha,\beta x}(\mathbf{X}) \\
& - \left\langle y_3\left(a_{ijkl}(\mathbf{y})\mathrm{N}_{k,ly}^{1\alpha\beta}(\mathbf{y}) - a_{ij\beta\alpha}(\mathbf{y})y_3 - e_{kij}\Phi_{,ky}^{1\alpha\beta}(\mathbf{y})\right)\right\rangle u_{3,\alpha x\beta x}^{(-1)}(\mathbf{X}) \\
& - \left\langle y_3\left(a_{ijkl}(\mathbf{y})\boldsymbol{N}_{k,ly}^{\alpha}(\mathbf{y}) - e_{kij}(\mathbf{y})\Phi_{,ky}^{\alpha}(\mathbf{y}) - e_{\alpha ij}(\mathbf{y})\right)\right\rangle \Theta_{,\alpha x}(\mathbf{X})
\end{aligned} \tag{5.29}
$$

The constitutive relations (5.27)–(5.29) can be written as follows (for $ij = \gamma\delta$ in Eqs. (5.27) and (5.29) and $i = \gamma$ in Eq. (5.28), $N_{\gamma\delta} = N_{\gamma\alpha}^{(-3)}$ and $M_{\gamma\delta} = M_{\gamma\delta}^{(-3)}$):

$$
\begin{aligned}
N_{\gamma\delta} &= A_{\gamma\delta\alpha\beta}^{0}V_{\alpha,\beta x} + A_{\gamma\delta\alpha\beta}^{1}u_{3,\alpha x\beta x}^{(-1)} + E_{\alpha\gamma\delta}^{n}\Theta_{,\alpha x} \\
E_{\gamma}^{(0)} &= E_{\gamma\alpha\beta}^{e}V_{\alpha,\beta x} + E_{\gamma\alpha\beta}^{1e}u_{3,\alpha x\beta x}^{(-1)} + E_{\gamma\alpha}\Theta_{,\alpha x} \\
M_{\gamma\delta} &= A_{\gamma\delta\alpha\beta}^{1}V_{\alpha,\beta x} + A_{\gamma\delta\alpha\beta}^{2}u_{3,\alpha x\beta x}^{(-1)} + E_{\gamma\delta\alpha}^{1m}\Theta_{,\alpha x}
\end{aligned} \tag{5.30}
$$

where

$$
\begin{aligned}
A_{\gamma\delta\alpha\beta}^{\nu+\mu} &= \left\langle (-1)^{\mu}y_3^{\mu}\left(a_{\gamma\delta kl}(\mathbf{y})\mathrm{N}_{k,ly}^{\nu\alpha\beta}(\mathbf{y}) + (-1)^{\nu}a_{\gamma\delta\beta\alpha}(\mathbf{y})y_3^{\nu} - e_{k\gamma\delta}(\mathbf{y})\Phi_{,ky}^{\nu\alpha\beta}(\mathbf{y})\right)\right\rangle \\
E_{\alpha\gamma\delta}^{n} &= \left\langle a_{\gamma\delta kl}(\mathbf{y})\mathbf{N}_{k,ly}^{\alpha}(\mathbf{y}) - e_{k\gamma\delta}(\mathbf{y})\Phi_{,ky}^{\alpha}(\mathbf{y}) - e_{\alpha\gamma\delta}(\mathbf{y})\right\rangle \\
E_{\gamma\alpha\beta}^{e} &= \left\langle \varepsilon_{\gamma k}(\mathbf{y})\Phi_{,ky}^{0\alpha\beta}(\mathbf{y}) + e_{\gamma\beta\alpha}(\mathbf{y}) + e_{\gamma kl}(\mathbf{y})\mathrm{N}_{k,ly}^{(0\alpha\beta)}(\mathbf{y})\right\rangle \\
E_{\gamma\alpha\beta}^{1e} &= \left\langle \varepsilon_{\gamma k}(\mathbf{y})\Phi_{,ky}^{1\alpha\beta}(\mathbf{y}) - e_{\gamma\beta\alpha}(\mathbf{y})y_3 + e_{\gamma kl}(\mathbf{y})\mathrm{N}_{k,ly}^{1\alpha\beta}(\mathbf{y})\right\rangle \\
E_{\gamma\delta\alpha}^{1m} &= -\left\langle y_3\left(a_{\gamma\delta kl}(\mathbf{y})\mathbf{N}_{k,ly}^{\alpha}(\mathbf{y}) - e_{k\gamma\delta}(\mathbf{y})\Phi_{,ky}^{\alpha} - e_{\alpha\gamma\delta}(\mathbf{y})\right)\right\rangle, \nu,\mu = 0,1 \\
E_{\gamma\alpha} &= \left\langle \varepsilon_{rj}(\mathbf{y})\Phi_{,jy}^{\alpha}(\mathbf{y}) + \varepsilon_{r\alpha} + e_{rkl}(\mathbf{y})\mathbf{N}_{k,ly}^{\alpha}(\mathbf{y})\right\rangle
\end{aligned} \tag{5.31}
$$

The boundary conditions are obtained by substituting Eqs. (5.17) and (5.18) into Eq. (4.8) for $k = 0$. These are

$$
V_{\alpha}(\mathbf{X}) = 0, \qquad u_3^{(-1)}(\mathbf{X}) = u_{3,\alpha x}^{(-1)}n_{\alpha}(\mathbf{X}) = 0, \qquad E^{(0)} = e(\mathbf{X}) \quad \text{on } \partial D,\ \alpha = 1,2 \tag{5.32}
$$

Here $n_{\alpha}(\alpha = 1,2)$ is a normal vector to ∂D.

The above obtained equations (5.30) represent constitutive relations of the two-dimensional homogenized plate model. The coefficients in these relations given by Eq. (5.31) are the effective stiffnesses and effective piezoelectric and dielectric coefficients of the homogenized plate. Eqs. (4.6) and (4.7) with $p = -2$ represent the equilibrium equations. It is possible to eliminate the shearing forces $N_{3\beta,\beta x}^{(-2)}$ as it is done in the classical plate theory. Relations (5.32) represent the boundary conditions. As a result, we obtained a closed-form problem for two-dimensional homogenized plate. It is important to note that not only effective properties, but also local mechanical and electrical fields can be determined using the above-derived homogenized piezoelectric plate model. These local fields can be calculated by means of the formulas (5.25) and (5.26) as soon as the two-dimensional homogenized problem is solved.

6. Symmetry of the effective coefficients of the homogenized piezoelastic plate

Material constants of a piezoelectric material have certain properties of symmetry or anti-symmetry. Namely, we have $-e_{kij}$ in Eq. (2.5) and e_{ikl} in Eq. (2.6). It has been shown that analog of this symmetry takes place for homogenized three-dimensional composite material, see Kalamkarov (1992). The governing equations for the homogenized piezoelastic plate are different because they also include the bending stiffnesses and curvatures. Let us examine now the symmetry properties of the effective piezoelectric coefficients of the homogenized plane model.

Multiplying the first equation from Eq. (5.5) by N_i^γ, and the second by Φ^γ, and integrating by parts over the unit cell P_1, taking into account boundary conditions (5.6) and periodicity conditions, and subtracting the obtained equations, we get

$$\left\langle a_{ijkl}(\mathbf{y})\mathbf{N}^{0\alpha\beta}_{k,ly}N^\gamma_{i,jy} - e_{kij}\Phi^{0\alpha\beta}_{,ky}N^\gamma_{i,jy} + a_{ij\alpha\beta}(\mathbf{y})y_3^v N^\gamma_{i,jy} - e_{ikl}(\mathbf{y})\mathbf{N}^{0k\alpha}_{k,ly}\Phi^\gamma_{,iy} - \varepsilon_{ij}\Phi^{0k\alpha}_{,jy}\Phi^\gamma_{,iy} - e_{ik\alpha}(\mathbf{y})y_3^v\Phi^\gamma_{,iy})\right\rangle = 0, \quad v = 0,1 \tag{6.1}$$

Multiplying the first equation from Eq. (5.7) by $N_i^{0\alpha\beta}$, second by $\Phi^{0k\alpha}$ and integrating by parts over the unit cell P_1, taking into account boundary conditions (5.8) and periodicity conditions, and subtracting the obtained equations, we get

$$\left\langle a_{ijkl}(\mathbf{y})N^\gamma_{k,ly}\mathbf{N}^{0\alpha\beta}_{i,jy} - e_{kij}\Phi^\gamma_{,ky}\mathbf{N}^{0\alpha\beta}_{i,jy} - e_{\gamma ij}\mathbf{N}^{0\alpha\beta}_{i,jy} - e_{ikl}(\mathbf{y})N^\gamma_{k,ly}\Phi^{0k\alpha}_{,iy} + \varepsilon_{ij}\Phi^\gamma_{,jy}\Phi^{0k\alpha}_{,iy} + \varepsilon_{i\gamma}\Phi^{0k\alpha}_{,iy})\right\rangle = 0 \tag{6.2}$$

Taking into account symmetry of a_{ijkl} and ε_{ij} in indices i and j, we obtain from Eqs. (6.1) and (6.2)

$$\left\langle a_{ij\alpha\beta}(\mathbf{y})N^\gamma_{i,jy} - e_{i\alpha\beta}(\mathbf{y})\Phi^\gamma_{,iy}\right\rangle = -\left\langle e_{\gamma ij}\mathbf{N}^{0\alpha\beta}_{i,jy} + \varepsilon_{i\gamma}\Phi^{0k\alpha}_{,iy}\right\rangle \tag{6.3}$$

In accordance with Eq. (5.31) for effective coefficients of homogenized plate,

$$\mathrm{E}^n_{\alpha\gamma\delta} = \left\langle a_{\gamma\delta kl}(\mathbf{y})N^\alpha_{k,ly}(\mathbf{y}) - e_{\alpha\gamma\delta}(\mathbf{y})\Phi^\alpha_{,ky}(\mathbf{y}) - e_{\alpha\gamma\delta}(\mathbf{y})\right\rangle \tag{6.4}$$

$$\mathrm{E}^e_{\alpha\gamma\delta} = \left\langle \varepsilon_{\gamma j}(\mathbf{y})\Phi^{0\alpha\beta}_{,jy}(\mathbf{y}) + e_{\gamma\alpha\beta}(\mathbf{y}) + e_{\gamma kl}(\mathbf{y})\mathbf{N}^{0\alpha\beta}_{k,ly}(\mathbf{y})\right\rangle \tag{6.5}$$

Taking into account symmetry of a_{ijkl} and ε_{ij} in indices i and j, we obtain from Eqs. (6.3)–(6.5) the following anti-symmetry relation for the effective in-plane piezoelectric coefficients:

$$\mathrm{E}^n_{\alpha\gamma\delta} = -\mathrm{E}^e_{\alpha\gamma\delta}$$

This relation is analogous to the anti-symmetry property $e_{kij} = -e_{ijk}$ for piezoelectric constants in Eqs. (2.5) and (2.6).

From the unit cell problems (5.7) and (5.8) and (5.23) and (5.24) follows an analog of formula (6.3) which can be written as

$$\left\langle y_3(a_{ij\alpha\beta}(\mathbf{y})N^\gamma_{i,jy} - e_{i\alpha\beta}(\mathbf{y})\Phi^\gamma_{,iy})\right\rangle = \left\langle e_{\gamma ij}\mathbf{N}^{0\alpha\beta}_{i,jy} + \varepsilon_{i\gamma}\Phi^{0k\alpha}_{,iy}\right\rangle \tag{6.6}$$

In accordance with Eq. (5.31) for the effective coefficients of homogenized plate

$$E^{1m}_{\gamma\delta\alpha} = -\left\langle y_3(a_{\gamma\delta kl}(\mathbf{y})N^\alpha_{k,ly}(\mathbf{y}) - e_{k\gamma\delta}(\mathbf{y})\Phi^\alpha_{,ky}(\mathbf{y}) - e_{\alpha\gamma\delta}(\mathbf{y}))\right\rangle \tag{6.7}$$

$$E^{1e}_{\alpha\gamma\delta} = \left\langle \varepsilon_{\alpha j}(\mathbf{y})\Phi^{1\gamma\delta}_{,jy}(\mathbf{y}) - e_{\alpha\gamma\delta}(\mathbf{y})y_3 + e_{\alpha kl}(\mathbf{y})\mathbf{N}^{1\gamma\delta}_{k,ly}(\mathbf{y})\right\rangle \tag{6.8}$$

Taking into account symmetry of a_{ijkl} and ε_{ij} in indices i and j, we obtain from Eqs. (6.6)–(6.8) the following anti-symmetry relation for the effective in-plane piezoelectric coefficients:

$$E^{1m}_{\alpha\gamma\delta} = -E^{1e}_{\alpha\beta\delta}$$

Using the above-obtained relations, the constitutive relations of the homogenized plate (5.30) can be written as follows:

$$\begin{aligned} \mathrm{N}_{\gamma\delta} &= A^{0}_{\gamma\delta\alpha\beta} V_{\alpha,\beta x} + A^{1}_{\gamma\delta\alpha\beta} u^{(-1)}_{3,\alpha x\beta x} + E_{\alpha\gamma\delta}\Theta_{,\alpha x} \\ M_{\gamma\delta} &= A^{1}_{\gamma\delta\alpha\beta} V_{\alpha,\beta x} + A^{2}_{\gamma\delta\alpha\beta} u^{(-1)}_{3,\alpha x\beta x} + E^{1}_{\alpha\delta\alpha}\Theta_{,\alpha x} \\ E^{(0)}_{\gamma} &= -E_{\alpha\beta\gamma} V_{\alpha,\beta x} - E^{1}_{\alpha\beta\gamma} u^{(-1)}_{3,\alpha x\beta x} + E_{\alpha\gamma}\Theta_{,\alpha x} \end{aligned} \tag{6.9}$$

where

$$E_{\alpha\gamma\delta} = E^{n}_{\alpha\gamma\delta} = -E^{e}_{\alpha\beta\gamma}, \qquad E^{1}_{\alpha\gamma\delta} = E^{1m}_{\alpha\gamma\delta} = -E^{1e}_{\alpha\gamma\delta}$$

The effective coefficients $E_{\alpha\gamma\delta}$ and $E^{1}_{\alpha\gamma\delta}$ can be calculated using any of the appropriate formulas given in Eq. (5.31).

In a similar manner, one can derive the following symmetry relation for the effective stiffnesses of the homogenized plate:

$$A^{0+1}_{\gamma\delta\alpha\beta} = A^{1+0}_{\gamma\delta\alpha\beta}$$

It is analogous to symmetry relation obtained earlier for the plates in the case of elastic problem, see Kalamkarov (1992).

It is interesting to compare the obtained piezoelectric and the elastic homogenized problems. The homogenized equilibrium equations and the boundary conditions (see Section 4) are similar for both piezoelectric and elastic problems. Let us compare the constitutive relations. If $e_{ijk} = 0$, then the unit cell problems (5.5) and (5.23) decouple. They take forms of decoupled elasticity unit cell problem with respect to the functions $N^{\nu\beta\alpha}$, (see Kalamkarov (1992)) and electrostatics unit cell problem with respect to the function Φ^{α}. The functions $\Phi^{0\alpha\beta}(\mathbf{y}) = \Phi^{1\alpha\beta}(\mathbf{y}) = 0$. As a result we obtain two decoupled two-dimensional problems: two-dimensional elastic plate model (which is similar to one presented in Kalamkarov (1992)) and two-dimensional model for a thin dielectric layer. The constitutive equations of the elastic plate are given by the first two equations from Eq. (6.9) and the constitutive relations for the dielectric layer are given by the third equation from Eq. (6.9). For both the models $E^{e}_{\alpha\beta\gamma} = E^{1}_{\alpha\gamma\delta} = 0$.

7. Anisotropic laminated piezoelectric plate

The above-obtained asymptotic homogenization piezoelastic plate model has a general character. It can be applied to the analysis of global and local deformation of different types of piezoelectric composite and reinforced plates. The type of reinforcement will define the shape of the surface S_{ε}, and the type of a composite material inhomogeneity will define the material parameter functions $a_{ijkl}(\mathbf{y})$, $e_{kij}(\mathbf{y})$ and $\varepsilon_{ik}(\mathbf{y})$. The local unit cell problems can be solved by means of the appropriate analytical or numerical techniques. Let us consider now a laminated plate as a practically important type of a composite piezoelastic plate. In the case of laminated plate all the material characteristics are functions in only one variable y_3 and surface S_{ε} is plane. In this case $P_1 = [0, 1]$ and the unit cell problems (5.5) and (5.23) can be written as follows:

$$\begin{aligned} &(a_{i3k3}(y_3)\mathrm{N}^{\nu\prime}_{k} - e_{3i3}\Phi^{\nu\prime} + a_{i3\beta\alpha}(y_3)y^{\nu}_{3})' = 0 \\ &(e_{3k3}(y_3)\mathrm{N}^{\nu\prime}_{k} + \varepsilon_{33}\Phi^{\nu\prime} + e_{i3\alpha}(y_3)y^{\nu}_{3})' = 0 \quad \text{in } [0, 1] \end{aligned} \tag{7.1}$$

with the corresponding boundary conditions obtained from Eqs. (5.6) and (5.24); $\nu = 0, 1$. Here prime denotes a derivative in y_3. In this section for simplicity we omit indices $\beta\alpha$ for N^{ν}_{3} and Φ^{ν}.

The problem (7.1) represents four ordinary differential equations with respect to four unknown functions. Consider the case $a_{i3k3} = 0$ for $i \neq k$ and $a_{\gamma3\beta\alpha} = 0$, $\varepsilon_{3\alpha3} = 0$. In this case $N^{\nu}_{1} = N^{\nu}_{2} = 0$ and

Eq. (7.1) is reduced to two differential equations with respect to functions N_3^{ν} and Φ^{ν}. Integrating Eq. (7.1), we obtain

$$a_{3333}(y_3)\mathrm{N}_3^{\nu\prime} - e_{333}(y_3)\Phi^{\nu\prime} + a_{33\beta\alpha}(y_3)y_3^{\nu} = 0, \quad e_{333}(y_3)\mathrm{N}_3^{\nu\prime} + \varepsilon_{33}(y_3)\Phi^{\nu\prime} + e_{3\beta\alpha}(y_3)y_3^{\nu} = 0 \tag{7.2}$$

Solution of Eq. (7.2) is the following:

$$\begin{aligned} N_3^{\nu\prime} &= -e_{3\beta\alpha}(y_3)y_3^{\nu}e_{333}(y_3)/\Delta(y_3) - a_{33\beta\alpha}(y_3)y_3^{\nu}\varepsilon_{33}(y_3)/\Delta(y_3) \\ \Phi^{\nu\prime} &= -e_{3\beta\alpha}(y_3)y_3^{\nu}a_{3333}(y_3)/\Delta(y_3) - a_{33\beta\alpha}(y_3)y_3^{\nu}e_{333}(y_3)/\Delta(y_3) \end{aligned} \tag{7.3}$$

where $\Delta(y_3) = a_{3333}(y_3)\varepsilon_{33}(y_3) + e_{333}(y_3))^2$. Then

$$\begin{aligned} N_3^{\nu\prime} &= (-a_{33\beta\alpha}(y_3)\varepsilon_{33}(y_3) - e_{3\beta\alpha}(y_3)e_{333}(y_3))y_3^{\nu}/\Delta(y_3) \\ \Phi^{\nu\prime} &= (-e_{3\beta\alpha}(y_3)a_{3333}(y_3) + a_{33\beta\alpha}(y_3)e_{333}(y_3))y_3^{\nu}/\Delta(y_3) \end{aligned} \tag{7.4}$$

In the case under consideration the formula (5.31) for $A_{\gamma\delta\alpha\beta}^{\nu+\mu}$ takes the form:

$$A_{\gamma\delta\alpha\beta}^{\nu+\mu} = \left\langle (-1)^{\mu}y_3^{\mu}(a_{\gamma\delta33}(y_3)N_3^{\nu\prime} + (-1)^{\nu}a_{\gamma\delta\beta\alpha}(y_3)y_3^{\nu} - e_{3\gamma\delta}\Phi^{\nu\prime})\right\rangle$$

Substituting Eq. (7.4) into this formula, we obtain the following formula for the effective stiffnesses of two-dimensional homogenized plate:

$$\begin{aligned} A_{\gamma\delta\alpha\beta}^{\nu+\mu} = {}& \langle (-1)^{\mu+\nu}y_3^{\nu+\mu}a_{\gamma\delta\alpha\beta}\rangle + \langle (-1)^{\mu+\nu}y_3^{\nu+\mu}\big(-a_{\gamma\delta33}a_{33\beta\alpha}\varepsilon_{33} - a_{\gamma\delta33}e_{3\beta\alpha}e_{333} - e_{3\gamma\delta}e_{3\beta\alpha}a_{3333} \\ & + e_{3\gamma\delta}a_{33\beta\alpha}e_{333}\big)/\Delta\rangle \end{aligned} \tag{7.5}$$

In Eq. (7.5) and in the sequel the argument y_3 may be omitted for simplicity, i.e.,

$$\Delta \equiv \Delta(y_3); \;\; e_{333} \equiv e_{333}(y_3), \text{etc.}$$

In the similar way we can solve the unit cell problems (5.7) and (5.8). Eq. (5.7) under above formulated conditions for local constants takes the form (we omit index α for N_3' and Φ' for simplicity)

$$(a_{3333}(y_3)N_3' - e_{333}(y_3)\Phi' - e_{\alpha33}(y_3))' = 0, \quad (e_{333}(y_3)N_3' + \varepsilon_{33}(y_3)\Phi' + \varepsilon_{3\alpha}(y_3))' = 0 \tag{7.6}$$

Eq. (7.6) yields

$$a_{3333}(y_3)N_3' - e_{333}(y_3)\Phi' - e_{\alpha33}(y_3)' = 0, \quad e_{333}(y_3)N_3' + \varepsilon_{33}(y_3)\Phi' + \varepsilon_{3_\alpha}(y_3)' = 0 \tag{7.7}$$

We obtain from Eq. (7.7)

$$\begin{aligned} N_3' &= e_{\alpha33}\varepsilon_{33}/\Delta - \varepsilon_{3\alpha}e_{333}/\Delta \\ \Phi' &= e_{\alpha33}e_{333}/\Delta - \varepsilon_{3\alpha}a_{3333}/\Delta \end{aligned} \tag{7.8}$$

In the case under consideration the formula for the effective piezoelectric coefficient $E_{\alpha\gamma\delta}^{1m}$ from Eq. (5.31) can be written as follows:

$$E_{\alpha\gamma\delta}^{1m} = \left\langle y_3^{\nu}(a_{\gamma\delta33}(y_3)\mathbf{N}_3'(y_3) - e_{3\gamma\delta}(y_3)\Phi^{\alpha\prime}(y_3) - e_{\alpha\gamma\delta}(y_3))\right\rangle$$

Substituting Eq. (7.8) into this formula, we obtain the following formula for the effective piezoelectric coefficients of two-dimensional homogenized plate:

$$E_{\alpha\gamma\delta}^{1m} = -\langle y_3^{\nu}e_{\alpha\delta\gamma}\rangle + \langle y_3^{\nu}(a_{\gamma\delta33}e_{33\alpha}\varepsilon_{33}/\Delta - a_{\gamma\delta33}e_{333}\varepsilon_{3\alpha}/\Delta + a_{3333}e_{3\gamma\delta}\varepsilon_{3\alpha}/\Delta + e_{3\gamma\delta}e_{333}e_{33\alpha}/\Delta)\rangle \tag{7.9}$$

In the considered case, formula for the effective dielectric coefficients $E_{\gamma\alpha}$ from Eq. (5.31) can be written as follows:

$$E_{\gamma\alpha} = \left\langle \varepsilon_{r3}(y_3)\Phi^{\alpha\prime}(y_3) + \varepsilon_{r\alpha} + e_{r33}(y_3)\mathbf{N}^{\alpha\prime}(y_3) \right\rangle$$

Substituting Eq. (7.8) into this formula, we obtain the following formula for the effective dielectric coefficients of two-dimensional homogenized plate:

$$E_{\alpha\gamma} = \langle e_{\gamma 33}e_{33\alpha}\varepsilon_{33}/\Delta - e_{\gamma 33}\varepsilon_{3\alpha}e_{333}/\Delta - \varepsilon_{\gamma 3}\varepsilon_{3\alpha}a_{3333}/\Delta - \varepsilon_{\gamma 3}e_{\alpha 33}e_{333}/\Delta \rangle \qquad (7.10)$$

8. Numerical example: piezoelectric laminated plate

In this section we will use the above-obtained general relations to calculate effective properties of a laminated piezoelectric plate of a specific structure. Let us consider a symmetric laminated plate of an overall thickness $h = 0.01$ m, formed from the following three layers: piezoelectric layer with 10% of overall thickness + elastic (passive) layer of 80% overall thickness + another piezoelectric layer with 10% of overall thickness, see Fig. 2. We will also assume that all layers are made of isotropic and homogeneous materials, and both piezoelectric layers are made of the same material.

Let us calculate the effective bending stiffness A^2_{1111} given by Eq. (7.5). It was shown in the above Sections 5 and 6 that the effective stiffnesses of homogenized piezoelastic plate generally depend on the local piezoelectric constants, see Eqs. (5.16) and (7.5). In order to evaluate this dependence we will assume in the considering example that no electric field is applied, i.e., $e(\mathbf{X}) = 0$. The dependence of the effective bending stiffness A^2_{1111} on local piezoelectric constants will be due only to a piezoelectric effect in bending deformation.

Material constants from the local constitutive relations (2.5) and (2.6) are given as follows:

$$a_{1111} = a_{3333} = (1-\nu)E/((1+\nu)(1-2\nu)), \quad a_{1133} = \nu E/((1+\nu)(1-2\nu))$$

$$e_{ijk} = a_{ijmn}\beta_{mnk}(\text{in piezoelectric layers}), \quad \beta_{mnk} = \beta\delta_{mk}\delta_{nk}(\text{here no summation in } k)$$

$$\varepsilon_{ij} = \varepsilon\delta_{ij}$$

We will distinguish material properties of the upper and lower piezoelectric layers by subscript 1, and the properties of the middle elastic layer by subscript 2.

Plots of the effective bending stiffness A^2_{1111} calculated according to Eq. (7.5) vs. piezoelectric constant β_1 are shown in Fig. 3 for five different material compositions. As follows from this figure, the effective bending stiffness depends indeed on the piezoelectric constant β_1. This dependence is evident in case (1), it is weaker in cases (3) and (5), and it can be neglected in cases (2) and (4) in which the dielectric constant of the piezoelectric layer ε_1 is much larger than the dielectric constant of the passive elastic layer ε_2. In the piezoelectric composites used nowadays these conditions are commonly satisfied, and therefore for these

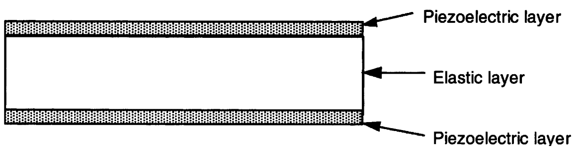

Fig. 2. Three-layer piezoelectric laminated plate.

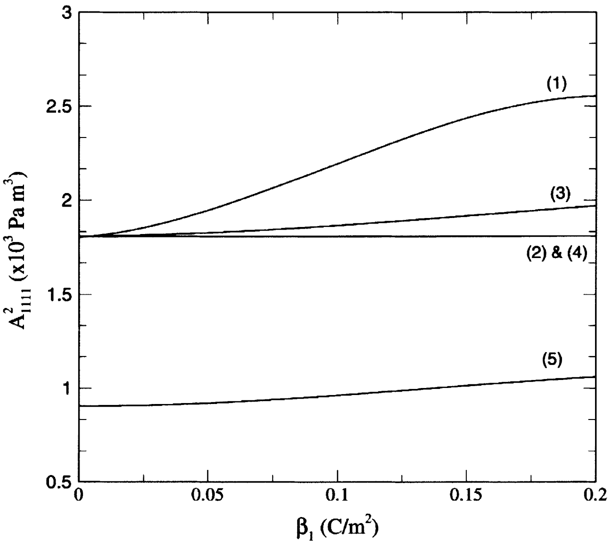

Fig. 3. Effective bending stiffness A^2_{1111} of the laminated piezoelastic plate shown in Fig. 2, calculated according to Eq. (7.5) vs. piezoelectric constant β_1 for the following material compositions: (1) $E_1 = 30$ GPa, $\nu_1 = 0.4$, $\varepsilon_1 = 1.0 \times 10^{-9}$ F/m, $E_2 = 10$ GPa, $\nu_2 = 0.3$, $\varepsilon_2 = 1.0 \times 10^{-7}$F/m, (2) $E_1 = 30$ GPa, $\nu_1 = 0.4$, $\varepsilon_1 = 1.0 \times 10^{-6}$ F/m, $E_2 = 10$ GPa, $\nu_2 = 0.3$, $\varepsilon_2 = 1.0 \times 10^{-9}$ F/m, (3) $E_1 = 10$ GPa, $\nu_1 = 0.4$, $\varepsilon_1 = 1.0 \times 10^{-9}$ F/m, $E_2 = 30$ GPa, $\nu_2 = 0.3$, $\varepsilon_2 = 1.0 \times 10^{-9}$ F/m, (4) $E_1 = 10$ GPa, $\nu_1 = 0.4$, $\varepsilon_1 = 1.0 \times 10^{-6}$ F/m, $E_2 = 30$ GPa, $\nu_2 = 0.3$, $\varepsilon_2 = 1.0 \times 10^{-9}$ F/m, (5) $E_1 = 10$ GPa, $\nu_1 = 0.4$, $\varepsilon_1 = 1.0 \times 10^{-9}$ F/m, $E_2 = 10$ GPa, $\nu_2 = 0.3$, $\varepsilon_2 = 1.0 \times 10^{-9}$ F/m. Material properties of the upper and lower piezoelectric layers (see Fig. 2) have subscript 1, and properties of the middle elastic layer have the subscript 2, overall plate thickness $h = 0.01$ m, $\beta_2 = 0$ in all cases.

materials the dependence of the effective stiffnesses on the piezoelectric constant may be neglected. In this case the following simplified formula can be used instead of Eq. (7.5):

$$A^2_{1111} = \langle E(y_3)y_3^2/(1 - \nu^2(y_3)) \rangle \tag{8.1}$$

Another approximation can be obtained using the following formula:

$$A^2_{1111} = A^0_{1111}h^2/12 \tag{8.2}$$

where h is the thickness of a laminated plate. Relation (8.2) is analogous to a formula which relates the bending and in-plane stiffnesses for homogeneous elastic plates. Note that this formula is valid for laminated plates formed by a large number of layers, see Kolpakov (1982). In the piezoelastic case formula (8.2) will provide dependence on β_1 through A^0_{1111} calculated using Eq. (7.5).

Plots in Figs. 4 and 5 compare results for effective bending stiffness A^2_{1111} calculated in accordance with formulas (7.5), (8.1) and (8.2) vs. piezoelectric constant β_1. Difference in cases considered in these two figures is only in the values of Poisson's ratio ν_1 of the piezoelastic layer: in the case of Fig. 4, $\nu_1 = 0.40$, and in the case of Fig. 5, $\nu_1 = 0.47$. Comparison of these figures demonstrates the influence of the value of Poisson's ratio of the piezoelastic material ν_1. Difference between results obtained using formulas (7.5) and (8.1) grows as ν_1 grows closer to the value of 0.5. For smaller value of Poisson's ratio ν_1 the results of Eqs. (7.5) and (8.1) are getting closer. It is also seen from Figs. 4 and 5 that results of formula (8.2) are not satisfactory for the three-layer laminated plate under consideration.

Fig. 6 shows dependencies of the effective bending stiffness A^2_{1111} calculated in accordance with formulas (7.5), (8.1) and (8.2) on larger values of piezoelectric constant β_1. It is seen from this figure that as β_1 grows

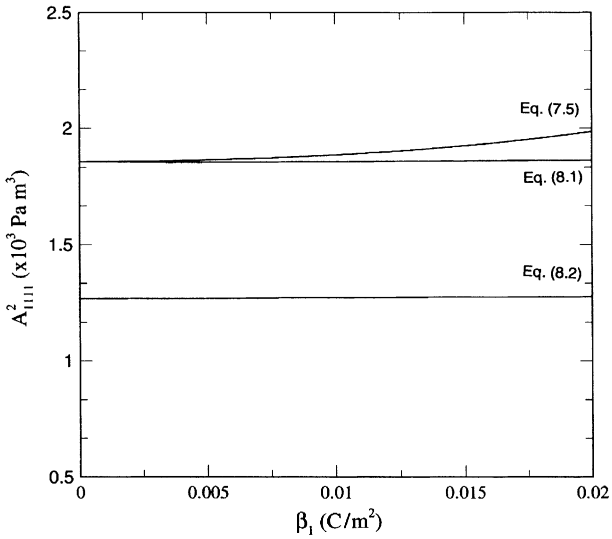

Fig. 4. Comparison of values of the effective bending stiffness A^2_{1111} of the laminated piezoelastic plate shown in Fig. 2, calculated according to Eqs. (7.5), (8.1) and (8.2) vs. piezoelectric constant β_1 for the following material composition: $E_1 = 30$ GPa, $v_1 = 0.45$, $\varepsilon_1 = 1.0 \times 10^{-9}$ F/m, $E_2 = 10$ GPa, $v_2 = 0.3$, $\varepsilon_2 = 1.0 \times 10^{-7}$ F/m.

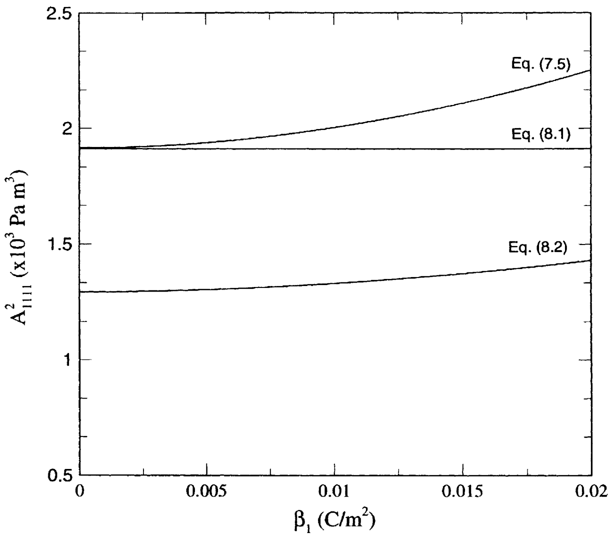

Fig. 5. Comparison of values of the effective bending stiffness A^2_{1111} of the laminated piezoelastic plate shown in Fig. 2, calculated according to Eqs. (7.5), (8.1) and (8.2) vs. piezoelectric constant β_1 for the following material composition: $E_1 = 30$ GPa, $v_1 = 0.47$, $\varepsilon_1 = 1.0 \times 10^{-9}$ F/m, $E_2 = 10$ GPa, $v_2 = 0.3$, $\varepsilon_2 = 1.0 \times 10^{-7}$ F/m.

the results of Eqs. (7.5) and (8.2) tend to constant values. These constant limits are achieved for β_1 approaching 0.4. This corresponds to piezoelastic deformation of about 40%, which is certainly too large for a linear problem. So, in practice, the effective bending stiffness will not reach a limit value.

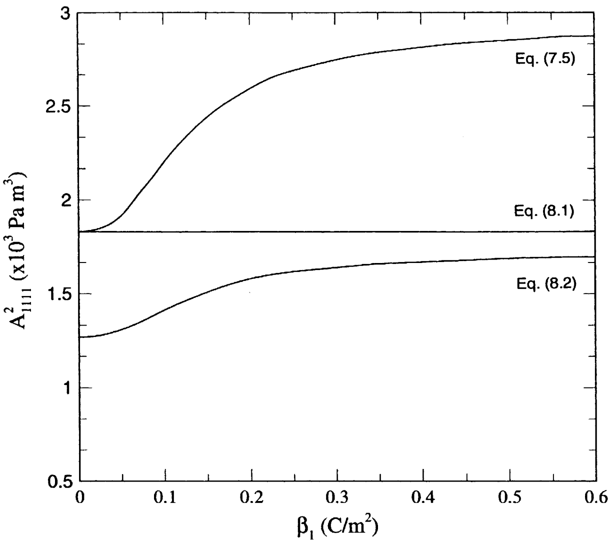

Fig. 6. Effective bending stiffness A_{1111}^2 of the laminated piezoelastic plate shown in Fig. 2, calculated in accordance with formulas (7.5), (8.1) and (8.2) for larger values of piezoelectric constant β_1. Material composition: $E_1 = 30$ GPa, $v_1 = 0.40$, $\varepsilon_1 = 1.0 \times 10^{-9}$ F/m, $E_2 = 10$ GPa, $v_2 = 0.3$, $\varepsilon_2 = 1.0 \times 10^{-7}$ F/m.

In the considered case $E_{\alpha\gamma\delta}^{1m} = 0$ due to the symmetry of the plate. As a result the elastic and the electric problems will decouple, and we have $E_{\gamma\alpha} = \langle \varepsilon_{r\alpha} \rangle$.

9. Conclusions

A new asymptotic homogenization piezoelastic composite plate model is obtained. Consideration is based on application of a modified two-scale asymptotic homogenization technique applied to a rigorously formulated piezoelectric problem for a three-dimensional thin composite solid of a periodic structure. The set of three-dimensional local unit cell problems, the constitutive relations and the governing equations for the homogenized anisotropic piezoelastic plate are derived. The obtained piezoelastic plate model makes it possible to determine both local mechanical and electrical fields, as well as the effective elastic, piezoelectric and dielectric properties, by means of solution of three-dimensional local unit cell problems and a global two-dimensional piezoelectric problem for a homogenized anisotropic plate. It is shown, in particular that the effective stiffnesses of the homogenized piezoelastic plate generally depend on the local piezoelectric constants of the material. This dependence is evaluated by a numerical example of a laminated piezoelastic plate. The general symmetry properties of the effective stiffnesses and piezoelectric coefficients of homogenized plate are derived. The general model is applied to a practically important case of a laminated anisotropic piezoelastic plate, for which the analytical formulas for the effective stiffnesses, piezoelectric and dielectric coefficients are obtained. Theory is illustrated by a numerical example of a three-layer piezoelastic laminated plate. In this example, the upper and lower layers are piezoelectric, while the middle layer is passive elastic. It is shown that the dependence of the calculated effective bending stiffness on the local piezoelectric constant of the material can be neglected in cases when the dielectric constant of the piezoelectric layers is much larger than the dielectric constant of the passive elastic layer. In this case a simplified formula for the effective bending stiffness is obtained. It is shown that this simplified formula provides satisfactory results for the effective bending stiffness when the value of Poisson's ratio of the piezoelectric material is relatively small. But for larger values of this Poisson's ratio closer to 0.5, the error of the simplified formula grows.

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